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Spacelike metric foliations

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Abstract

In this paper, we investigate spacelike metric foliations in lightlike complete spacetimes. When such a foliation satisfies the strong energy condition $\mathrm{Ric}^{\mathrm{v}}(e) \geq 0$ for timelike vectors e, it must be totally geodesic, and the metric is of higher rank, in the sense that each point of the spacetime is contained inside a flat, totally geodesic, timelike rectangle. If in addition $\mathrm{Ric}^{\mathrm{v}}(e) = 0$, then the metric is (at least locally) a product metric, with the leaves of the foliation tangent to one of the factors. © 1999 Elsevier Science B.V. All right reserved.

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1. Introduction

A foliation (without singular focal loci) of a Riemannian manifold is said to be *metric* if its leaves are locally everywhere equidistant. Equivalently, the leaves of $\mathcal{F}$ coincide, at least locally, with fibers of Riemannian submersions. Such foliations play a key role in the geometry of nonnegatively curved manifolds, cf. [2,4,6]. They are also significant in general relativity: One of the oldest singularity results [7] states that a geodesic, irrotational observer field in a spacetime M that satisfies the strong energy condition ($\mathrm{Ric}(x) \geq 0$ for timelike x) is either incomplete, or else M splits as a metric product.

The above result may be rephrased as follows: Any metric foliation by spacelike hypersurfaces of a timelike complete spacetime M satisfying the strong energy condition splits; i.e., M is locally isometric to a product $(I,\ \mathrm{d}t^2) \times (F, g)$ where I is an interval, and (F, g) is a Riemannian manifold. The leaves of the foliation are the sets $\{t\} \times F$, $t \in I$. Thus, for

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example, warped product cosmological models such as the Robertson–Walker spacetime are necessarily timelike incomplete.

Hypersurface foliations, however, have integrable orthogonal complement – the integral curves of the geodesic observer field. A more complicated and mathematically interesting model is that of a metric foliation by spacelike surfaces. Just as in the hypersurface case, the leaves are locally given by fibers of a semi-Riemannian submersion [4], but in this case, the orthogonal complement is no longer necessarily an integrable distribution. These foliations also turn out to be interesting from a physical point of view: In [8], causality of such spacetimes was shown to be well-behaved, and a large class of examples that are globally hyperbolic and satisfy the strong energy condition were constructed.

In this paper, we establish a rigidity result similar to the singularity one described above for hypersurfaces. In the codimension 1 case, the strong energy condition says that for x orthogonal to a leaf, the *vertical Ricci curvature* – i.e., the trace of the self-adjoint curvature transformation on the leaf given by $R(\cdot, x)x$ – is nonnegative. In the codimension 2 case studied here, we show that if the timelike vertical Ricci curvature is nonnegative, and if M is lightlike complete, then the foliation is totally geodesic. Moreover, each point in the spacetime is contained inside a flat totally geodesic rectangle. The argument is based on the study of a Raychaudhuri-type equation for horizontal geodesics which generalizes the classical one. We next discuss some important special cases: For example, if the vertical Ricci curvature is zero, then the foliation splits in the sense above, provided M is lightlike complete.

2. Spacelike metric foliations

We begin by recalling some basic local properties of semi-Riemannian submersions and foliations. For further details the reader is referred to [4,5,8]. Let $\mathcal{F}$ denote a spacelike metric foliation by surfaces of a spacetime M. $\mathcal{F}$ induces an orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of M into horizontal and vertical subbundles, with $\mathcal{V}$ tangent to the leaves. We write $e = e^{\mathrm{h}} + e^{\mathrm{v}}$ for the corresponding decomposition of $e \in TM$. If U is an open set in M, and $\pi : U \to B$ is a local submersion whose fibers coincide with the leaves of $\mathcal{F}$, then the condition that the foliation be metric means that

$$\langle \pi_* e, \pi_* e \rangle = \langle e^{\mathrm{h}}, e^{\mathrm{h}} \rangle, \qquad e \in TM.$$

The local geometry of $\mathcal{F}$ is determined by two tensor fields: The O'Neill tensor A is the (2,1) tensor field on $\mathcal{H}$ with values in $\mathcal{V}$

$$A_X Y = \overset{\mathrm{v}}{\nabla}_X Y = \tfrac{1}{2}[X, Y]^{\mathrm{v}}. \tag{1}$$

Thus, the vanishing of A is equivalent to the distribution $\mathcal{H}$ being integrable. The second fundamental tensor of $\mathcal{F}$ is the horizontal 1-form S with values in self-adjoint transformations of $\mathcal{V}$ given by

$$S_X V = -\overset{\mathrm{v}}{\nabla}_V X. \tag{2}$$

Thus, the restriction of S to a leaf is merely the second fundamental form of the leaf, and S vanishes along a leaf iff that leaf is totally geodesic.

A vector field X is said to be *basic* if it is the horizontal lift of some vector field on the local quotient manifold determined by $\mathcal{F}$. $[X, V]$ is always vertical for basic X and vertical V, and if $A_X^* : \mathcal{V} \to \mathcal{H}$ denotes the adjoint of A_X, then

$$\overset{\mathrm{h}}{\nabla}_V X = \overset{\mathrm{h}}{\nabla}_X V = -A_X^* V, \tag{3}$$

provided X is not lightlike. A geodesic γ that is horizontal at one point is everywhere horizontal, and induces local diffeomorphisms f^t between neighborhoods of $\gamma(0)$ in the leaf and corresponding neighborhoods of $\gamma(t)$ by horizontally lifting local projections of γ. The derivative of these "holonomy displacements" is given by $f_*^t u = J(t)$, where J is a nowhere zero Jacobi field along γ with $J(0) = u$. One can now give a Lorentzian version of a corresponding result for Riemannian manifolds, cf. also [9]:

Lemma. *Along a nonnull horizontal geodesic γ, one has the Riccati-type equation*

$$S_{\dot\gamma}'^{\,\mathrm{v}} = S_{\dot\gamma}^2 - A_{\dot\gamma} A_{\dot\gamma}^* + R_{\dot\gamma}^{\mathrm{v}}, \quad \text{where} \quad R_{\dot\gamma}^{\mathrm{v}} := R^{\mathrm{v}}(\cdot, \dot\gamma)\dot\gamma. \tag{4}$$

Proof. Consider a holonomy Jacobi field J as above. By (2) and (3),

$$J' = -A_{\dot\gamma}^* J - S_{\dot\gamma} J. \tag{5}$$

Thus, if T is a vertical vector field along γ, then

$$\begin{aligned}
\langle R(T, \dot\gamma)\dot\gamma, J\rangle &= -\langle T, J''\rangle = \langle T, (A_{\dot\gamma}^* J)'^{\mathrm{v}}\rangle + \langle T, (S_{\dot\gamma} J)'^{\mathrm{v}}\rangle \\
&= \langle T, A_{\dot\gamma} A_{\dot\gamma}^* J\rangle + \langle T, S_{\dot\gamma} J\rangle' - \langle T'^{\mathrm{v}}, S_{\dot\gamma} J\rangle \\
&= \langle A_{\dot\gamma} A_{\dot\gamma}^* T, J\rangle + \langle S_{\dot\gamma} T, J\rangle' - \langle S_{\dot\gamma}(T'^{\mathrm{v}}), J\rangle \\
&= \langle A_{\dot\gamma} A_{\dot\gamma}^* T, J\rangle + \langle (S_{\dot\gamma} T)'^{\mathrm{v}}, J\rangle - \langle S_{\dot\gamma}^2 T, J\rangle - \langle S_{\dot\gamma}(T'^{\mathrm{v}}), J\rangle.
\end{aligned}$$

Rearranging terms,

$$\langle (S_{\dot\gamma} T)'^{\mathrm{v}} - S_{\dot\gamma}(T'^{\mathrm{v}}), J\rangle = \langle (S_{\dot\gamma}^2 - A_{\dot\gamma} A_{\dot\gamma}^* + R_{\dot\gamma}^{\mathrm{v}})T, J\rangle.$$

Since for any t_0, there exist holonomy fields J_i that form an orthonormal basis of the vertical space at $\gamma(t_0)$, the lemma follows. □

For horizontal $e \in T_pM$, the *vertical Ricci curvature* of e is defined to be $\mathrm{Ric}^{\mathrm{v}}(e) = \sum_i \langle R(v_i, e)e, v_i\rangle$, where $\{v_i\}$ denotes an orthonormal basis of the vertical space at p. $\mathcal{F}$ is said to satisfy the *strong energy condition* if $\mathrm{Ric}^{\mathrm{v}}(e) \geq 0$ for all timelike e.

Theorem. *If $\mathcal{F}$ satisfies the strong energy condition and M is lightlike complete, then $\mathcal{F}$ is totally geodesic. Moreover, M is foliated by "flats", in the sense that every point of M is contained inside a flat, totally geodesic timelike rectangle.*

Proof. For lightlike horizontal e, the adjoint A_e^* of $A_e : \mathcal{H} \cap e^\perp \to \mathcal{V}$ is not well defined, because $\mathcal{H} \cap e^\perp$ is a degenerate subspace. Nevertheless, $A_e A_e^* : \mathcal{V} \to \mathcal{V}$ *is* well defined, because $A_e e = 0$, so that A_e may be considered to have the set $\{x \in \mathcal{H} \cap e^{\perp} \mid x \text{ is spacelike}\}$ as domain. Thus, for a codimension 2 spacelike foliation, $A_e A_e^* = 0$.

Now, let γ be lightlike, and set $s = \frac{1}{2} \operatorname{tr} S_{\dot\gamma}$, $B = S_{\dot\gamma} - s \operatorname{Id}$, so that B is the trace-free part of $S_{\dot\gamma}$. With the standard inner product $\langle B, C\rangle = \operatorname{tr} BC$ on the space of self-adjoint operators on $\mathcal{V}$, we have $\|S_{\dot\gamma}\|^2 = \|B\|^2 + 2s^2$. Taking traces in the Riccati equation (4) then yields

$$s' = s^2 + \tfrac{1}{2}\|B\|^2 + \operatorname{Ric}^{v}(\dot\gamma).$$

It follows that if γ is defined for all parameter values, then s, B, and $\operatorname{Ric}^{v}(\dot\gamma)$ all vanish. Consequently, $S_{\dot\gamma} \equiv 0$. Given a parameter value t_0, decompose $e = \dot\gamma(t_0)$ into a sum $x + y$ of spacelike x and timelike y, with $\|x\|^2 = -\langle y, y\rangle$, and $\langle x, y\rangle = 0$. Then $x \pm y$ is lightlike, so that $0 = S_{x \pm y} = S_x \pm S_y$. Thus, $S \equiv 0$; i.e., the leaves are totally geodesic.

We now proceed to establish the existence of flats in M. If γ denotes any timelike horizontal geodesic, then the image of $A_{\dot\gamma}$ is one-dimensional; i.e., $A_{\dot\gamma}^*$ has nontrivial kernel. Thus, for any two holonomy fields J_i along γ, the derivatives J_i' are linearly dependent at each point, since $J_i' = -A_{\dot\gamma}^* J_i$ by (5) and the theorem. Write $f J_1' + h J_2' = 0$ for some functions f and g, and differentiate to obtain

$$(f' J_1' + g' J_2') + (f J_1'' + g J_2'') = 0.$$

The expression inside the first set of parentheses is horizontal, whereas the one inside the second set is always vertical since by (5), $J_i'' = -R(J_i, \dot\gamma)\dot\gamma = -A_{\dot\gamma} A_{\dot\gamma}^* J_i$. Thus, both expressions vanish, and $f J_1' + g J_2' = f' J_1' + g' J_2' = 0$. This implies that f and g differ by a multiplicative constant; i.e., $a J_1' + b J_2' = 0$ for some numbers a and b. But then $J = a J_1 + b J_2$ is a *parallel* Jacobi field along γ, and in particular, $R(J, \dot\gamma)\dot\gamma = 0$.

So far, we have shown the existence of infinitesimal flat rectangles along timelike geodesics. In order to conclude the argument, we claim it suffices to establish the existence of a flat through any point where the A-tensor does not vanish: Indeed, if the set K of points where A vanishes contains an open set U, then the foliation splits locally isometrically over U; i.e., U decomposes as a metric product with the leaves of $\mathcal{F}$ tangent to one of the factors (and one then trivially obtains flats). This is an easy consequence of de Rham's holonomy theorem [1]: The fact that both A and S vanish over U means that the vertical distribution $\mathcal{V}$ is invariant under parallel translation. So consider a point p where A is nonzero, and a local unit vector field T that spans the kernel of A^*. From the arguments in the previous paragraph, the restriction of T along any horizontal timelike geodesic is a parallel Jacobi field. It now only remains to show that T is totally geodesic, for then the rectangle $(t, s) \mapsto \exp_{\gamma(t)} sT$ is flat and totally geodesic. But since the leaves are totally geodesic, $A_X Y$ is a Killing field along leaves [1], so that

$$\langle \nabla_T T, A_X Y\rangle(p) = -\langle T, \nabla_T A_X Y\rangle(p) = 0.$$

Thus, T is a totally geodesic field along each leaf. But since leaves are themselves totally geodesic in M, the claim follows. □

Remark. *The argument above goes through word for word in the case of a spacelike codimension 2 metric foliation in a Lorentzian manifold of arbitrary dimension, provided the foliation satisfies the strong energy condition.*

Corollary. *Let $\mathcal{F}$, M be as in Theorem 1. If* $\mathrm{Ric}^{\mathrm{v}}(e) = 0$ *for horizontal timelike e, then $\mathcal{F}$ splits; i.e., M decomposes locally as a metric product, with the leaves of $\mathcal{F}$ coinciding with one of the factors.*

Proof. In order to establish the splitting, it suffices, as observed in the proof of the Theorem, to show that $A \equiv 0$. But from (4) and the fact that S is identically zero, we deduce that for a timelike geodesic γ, $R(v, \dot{\gamma})\dot{\gamma} = A_{\dot{\gamma}} A^*_{\dot{\gamma}} v$ for vertical v, so that $\langle R(v, \dot{\gamma})\dot{\gamma}, v\rangle = \|A^*_{\dot{\gamma}} v\|^2 \geq 0$. Thus, if $\mathrm{Ric}^{\mathrm{v}}(\dot{\gamma}) = 0$, the A-tensor must vanish. □

Example. The Corollary implies that the only codimension 2 spacelike metric foliations of Minkowski space are congruent to metric products. This contrasts with the Euclidean case, where such foliations need not, in general, split: Consider for example the free $\mathbb{R}^2$-action on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ given by

$$(s, t)(p, q) = (e^{is} p, q + (s, t)), \qquad (s, t) \in \mathbb{R}^2, \quad (p, q) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Since this action is by isometries, the leaves are equidistant, and the orbit foliation is metric; it does not, however, split. In fact, the leaves are ruled surfaces, and the only totally geodesic leaf is the one that passes through the origin, cf. also [3].

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